The Mathematics of Flat Parachutes

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1.0 Introduction

The prevailing device for model rocket recovery is by far the ubiquitous parachute. Parachutes for model rocketry purposes are available in a broad range of sizes and in a rainbow of colors, and are made from a variety of different materials. However, the modeler may choose to make his own parachute, sometimes to save the cost of commercial parachutes, but more often because his project requires a non-standard size. This may be the case for competition or for special payload models, where a particular non-standard diameter is needed for a duration event or a certain rate of descent.

Perhaps the most complete shape for a parachute is a hemisphere. Many rocketeers will recall that this was the shape of the parachutes used in the Space Program, successfully delivering manned payloads to rest in the ocean for subsequent sea recovery operations. While hemispherical parachutes function very well, they can be complex to make, as the shape is 3 dimensional. Making a hemispherical shape requires the modeler to cut pieces of material into special curved segmental shapes, called gores, which when fitted together will form the hemisphere.

Fortunately hemispherical parachutes aren't really needed in most model rocket applications. Most commonly available parachutes are in fact created from standard two-dimensional (i.e.: flat) geometric figures, such as hexagons or octagons. When billowed from the weight of a model, these parachutes do a very good job approximating the performance of a hemispherical parachute. More importantly, because these parachutes are derived from flat figures, they are easier for the modeler to prepare from scratch.

This paper explores the polygonal geometry that characterizes the conventional two-dimensional (i.e.: flat) parachute that most model rocketeers use. A general solution is developed that will permit one to calculate the size (diameter) of the parachute needed to deliver a required canopy area.

2.0 Parachute Considerations

Before examining the geometry associated with flat parachutes, the first question that needs to be answered is - How big of a parachute do I need? The rate of descent will be dependent on the area of the parachute; once we know the required minimum area, a little geometry will tell us the diameter (size) we need to make the parachute.

In his book, "Model Rocket Design and Construction", 2^{nd} Edition, Tim Van Milligan (Apogee Components) provides a useful formula for calculating the minimum parachute area needed for a safe descent speed for a given model rocket mass. The formula is given as:

$$A_P = \frac{2gm}{\rho C_d V^2}$$

Where:

g= the acceleration due to gravity, 9.81 m/s^2 at sea level

m = the mass of the rocket (propellant consumed)

ρ = the density of air at sea level (1225 g/m^3)

C_d= the coefficient of drag of the parachute – estimated to be 0.75 for a round canopy

V = the descent velocity of the rocket, 11 to 14 ft/s (3.35 m/s to 4.26 m/s) being considered a safe descent speed.

With this descent rate equation, and a good calculator, one can readily find the needed minimum parachute area for a particular model or mission. To determine its size (diameter), we must generate an expression that relates area to size, and we must take into consideration the shape we choose for the parachute, as shape and diameter will dictate available surface area.

3.0 Parachute Geometry

Let's inscribe an n-sided polygon inside a circle. As an inscribed polygon, its vertices will be tangent to the circle, and the distance from its center to any vertex will be r, the radius of the circle. Figure 3-1 illustrates what an inscribed polygon looks like; in this case we have chosen to inscribe a regular octagon inside the circle.

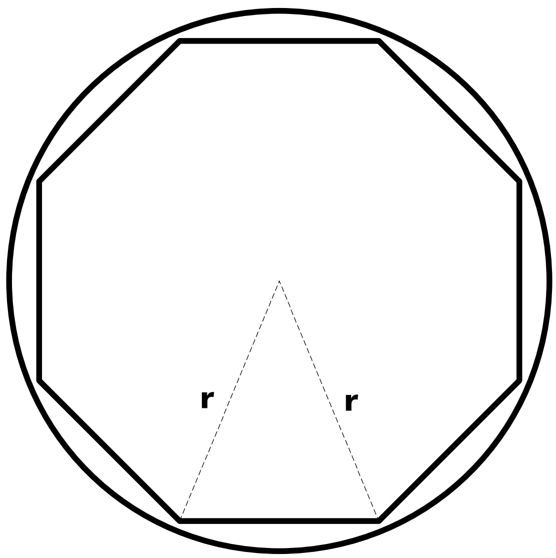

Figure 3-1: Inscribed Polygon

In this illustration two lines are shown, each originating from the center of the circle and extending to a vertex; together they form an isosceles triangle, the triangle having two identical sides r, and a base the length of the polygons' side. If similar lines were drawn to each remaining vertex, we would readily see that the octagon is made up of 8 identical isosceles triangles. We can extend this principle generally and say that an n-sided polygon is made up of n identical triangles, each triangle corresponding to one of the polygons' sides.

We can also see that the area of the polygon is just the sum of the areas of the triangles that comprise it; here, in the case of this particular polygon, its area is equal to 8 times the area of one triangle, or:

$A_O = 8A_T$, where A_O is the area of the octagon, and A_T is the area of the triangle.

Generalizing for any n-sided polygon, our expression for area is:

$A_P = nA_T$

With this concept now established, a little geometry will permit us to calculate the dimensions of our parachute. To do this, we simply need to establish the area of the elemental triangle that makes up the polygon, and then use the relationship above to calculate the total area.

Let's extract this triangle from the polygon and take a closer look at it:

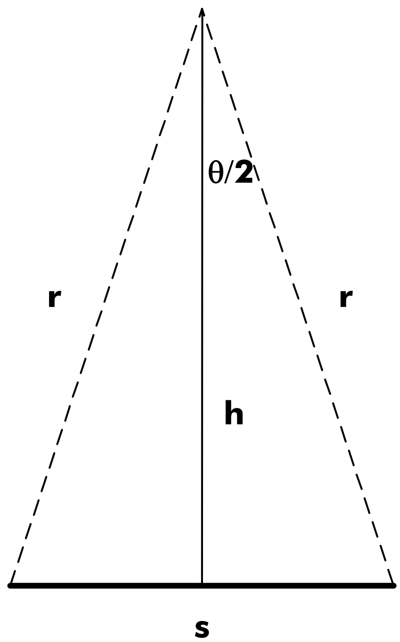

Figure 3-2: The Elemental Triangle

We know that the area of this triangle is:

$$A_T = \frac{s}{2} \bullet h = \frac{sh}{2}$$

We will now manipulate this relationship so that it can be expressed entirely in terms of r. To do this, we will use some trigonometry.

As reasoned earlier, an n-sided polygon is made up of n identical, elemental triangles. The angle subtended at the triangle's apex is θ; since there are n triangles making up the polygon, the value of θ must be 360°/n. For our purposes, we are interested in the angle between r and h; since this is half of θ (remember, we are dealing with an isosceles triangle), its value must be 360°/2n, or 180°/n once reduced.

We can derive the following expressions from the characteristics of the triangle:

$$\frac{s/2}{r} = \frac{s}{2r} = \sin\left(\frac{\theta}{2}\right) = \sin\left(\frac{180°}{n}\right);\ \therefore s = 2r\sin\left(\frac{180°}{n}\right)$$

and

$$\frac{h}{r} = \cos\left(\frac{180°}{n}\right);\ \therefore h = r\cos\left(\frac{180°}{n}\right)$$

Recalling that the area of the triangle is $A_T = \frac{sh}{2}$, we can then make the substitutions for s and h and get:

$$A_T = \frac{2r\sin\left(\frac{180°}{n}\right) \bullet r\cos\left(\frac{180°}{n}\right)}{2} = r^2 \sin\left(\frac{180°}{n}\right)\cos\left(\frac{180°}{n}\right)$$

There is a trigonometric identity that can be used to further reduce this expression, as follows:

For an angle α, $\sin 2\alpha = 2\sin\alpha\cos\alpha$

Here, if we set $\alpha = \frac{180°}{n}$, then $\sin 2\alpha = \sin\left(\frac{360°}{n}\right) = 2\sin\left(\frac{180°}{n}\right)\cos\left(\frac{180°}{n}\right)$

Applying this identity to our expression for A_T, we get:

$$A_T = \frac{r^2}{2} \bullet 2\sin\left(\frac{180°}{n}\right)\cos\left(\frac{180°}{n}\right) = \frac{r^2}{2}\sin\left(\frac{360°}{n}\right)$$

Substitute this result into our expression for A_P, and the area of our n-sided polygon becomes

$$A_P = \frac{nr^2}{2}\sin\left(\frac{360°}{n}\right)$$

We now have a general expression in terms of the radius, r, which we can use to calculate the area of any n-sided polygon.

Typically, we measure the diameter of a parachute as opposed to its radius, so with a little more algebra, we can transform the result into one expressed in terms of diameter.

Recall that $r = \frac{d}{2}$, where d is the diameter of the parachute/circle:

Then $A_P = \frac{n\left(\frac{d}{2}\right)^2}{2}\sin\left(\frac{360°}{n}\right) = \frac{nd^2}{8}\sin\left(\frac{360°}{n}\right)$

For practical purposes, we would calculate the required parachute area for a particular model from the descent rate equation. Once we know the area, we can use the expression from above to determine the required diameter, depending on the type of the parachute we intend to make (hexagonal, octagonal, or other).

Rearranging our equation, we can solve for d:

$$d = 2\sqrt{\frac{2A_P}{n\sin\left(\frac{360°}{n}\right)}}$$

We can complete the exercise definitively by substituting the descent rate equation for parachute area in the place of A_P; then we get:

$$d = 2\sqrt{\frac{4gm}{n\rho C_d V^2 \sin\left(\frac{360°}{n}\right)}}$$

Let's work out some practical examples. For a hexagonal parachute, we know $n = 6$.

So plugging 6 in for n, we get:

$$\therefore d \cong 1.2408\sqrt{A_P}$$

For an octagonal parachute, we know n=8:

$$\therefore d \cong 1.1892\sqrt{A_P}$$

Why should it make sense for the coefficient (the multiplier) of the parachute area to be smaller for an octagon? Well, if we recall Figure 3-1, it can be readily seen that the area of an octagon will be larger (cover more of the circle) than that of a hexagon for the same radius. So to arrive at the same parachute area, the diameter of a hexagonal parachute will need to be larger than that of an octagonal one.

4.0 Other Ways of Looking at the Same Thing

In the previous section, we derived an expression that related parachute diameter to the parachute's shape and area. Finding the diameter is important, as this parameter is the most useful one for laying out the parachute.

However, with some further algebraic manipulation we can re-work the result we found to express the area of the parachute in terms of the size of its sides, and also in terms of the distance measured from one side across to an adjacent side. These re-worked expressions provide an alternate way of calculating the area of a known parachute.

Let's go back and look at the Elemental Triangle:

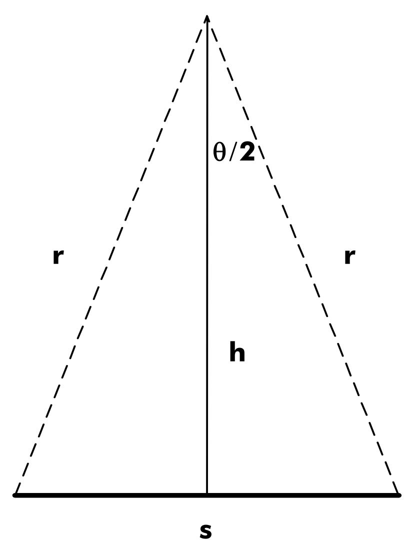

Figure 4-1: The Elemental Triangle

Recall that $\frac{s}{2r} = \sin\left(\frac{\theta}{2}\right) = \sin\left(\frac{180°}{n}\right)$

Then $r = \dfrac{s}{2\sin\left(\frac{180°}{n}\right)}$

Recall that $A_T = r^2 \sin\left(\frac{180°}{n}\right)\cos\left(\frac{180°}{n}\right)$

Substituting for r:

$$A_T = \frac{s^2}{4\sin^2\left(\frac{180°}{n}\right)}\sin\left(\frac{180°}{n}\right)\cos\left(\frac{180°}{n}\right)$$

Reducing gives:

$$A_T = \frac{s^2\cos\left(\frac{180°}{n}\right)}{4\sin\left(\frac{180°}{n}\right)} = \frac{s^2}{4\tan\left(\frac{180°}{n}\right)}$$

$$\therefore A_P = nA_T = \frac{ns^2}{4\tan\left(\frac{180°}{n}\right)}$$

This result gives us the area of the parachute expressed in terms of the length of a side.

Let's re-express the relation in terms of the distance from one side to its opposing side – let's call this distance D.

Then $D = 2h\,;\ \therefore h = \frac{D}{2}$

$$\frac{h}{r} = \cos\left(\frac{180°}{n}\right)$$

$$r = \frac{h}{\cos\left(\frac{180°}{n}\right)} = \frac{D}{2\cos\left(\frac{180°}{n}\right)}$$

Substituting this value for r into the equation for A_T, we get:

$$A_T = \frac{D^2}{4\cos^2\left(\frac{180°}{n}\right)}\sin\left(\frac{180°}{n}\right)\cos\left(\frac{180°}{n}\right) = \frac{D^2\sin\left(\frac{180°}{n}\right)}{4\cos\left(\frac{180°}{n}\right)} = \frac{D^2}{4}\tan\left(\frac{180°}{n}\right)$$

$$\therefore A_P = nA_T = \frac{nD^2}{4}\tan\left(\frac{180°}{n}\right)$$

This result gives us the area of the parachute expressed in terms of the distance measured from side to opposite side. Note that this expression is only valid for polygons with an even number of sides.

5.0 Conclusions & Findings

This paper demonstrates several ways to determine the area of a parachute depending on the parameters available. It provides a formula for determining the minimum diameter needed to provide a parachute canopy of prescribed area, a calculation that is important if the modeler intends to make the parachute himself.

The following summarizes the findings of this analysis:

5.1 Minimum Parachute Area for a Given Model

$$A_P = \frac{2gm}{\rho C_d V^2}$$

(Reference "Model Rocket Design and Construction", 2nd Edition, Tim Van Milligan (Apogee Components)).

5.2 Diameter of a Parachute, given its Area

$$d = 2\sqrt{\frac{2A_P}{n\sin\left(\frac{360^\circ}{n}\right)}}$$

n = the number of sides of the parachute.
A_P= the area of the parachute.

For a Hexagonal parachute, $d \cong 1.2408\sqrt{A_P}$

For an Octagonal parachute, $d \cong 1.1892\sqrt{A_P}$

5.3 Area of a Parachute, given its Diameter

$$A_P = \frac{nd^2}{8}\sin\left(\frac{360^\circ}{n}\right)$$

n = the number of sides of the parachute.
d = the diameter of the parachute, measured vertex to vertex.

For a Hexagonal parachute, $A_H \cong (0.6495)d^2$

For an Octagonal parachute, $A_O \cong (0.7071)d^2$

5.4 Area of a Parachute, given the length of a Side

$$A_P = \frac{ns^2}{4\tan\left(\frac{180°}{n}\right)}$$

s = the length of a side.

For a Hexagonal parachute, $A_H \cong (2.5981)s^2$

For an Octagonal parachute, $A_O \cong (4.8284)s^2$

5.5 Area of a Parachute, given the distance between opposite sides

$$A_P = \frac{nD^2}{4}\tan\left(\frac{180°}{n}\right)$$

D = the distance between opposite sides, and n must be even.

For a Hexagonal parachute, $A_H \cong (0.8667)D^2$

For an Octagonal parachute, $A_O \cong (0.8284)D^2$

Appendix A: An Explicit Derivation for the Hexagonal Parachute

We can cross check the correctness of the general expression by looking at the characteristics of a hexagonal parachute. For this parachute, there will be 6 sides and it will be comprised of 6 elemental triangles. Unique in this case is the fact that all of the interior angles of each triangle are of the same value. Since two of the triangle sides are known to be equal to r, and with all angles equal, we can safely reason that the third side, the base, must also be equal to r. This leads to the conclusion that the elemental triangle in this case is an equilateral triangle. Figure A-1 illustrates this elemental triangle:

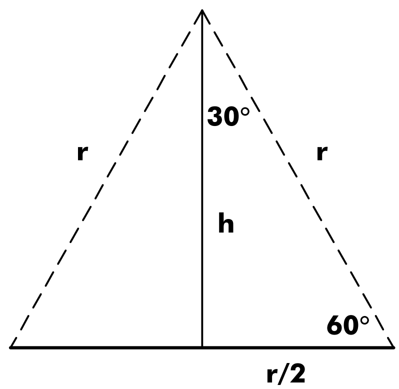

Figure A-1: Elemental Triangle from a Hexagon

Using the Pythagorean Theorem, we can draw the following relationships:

$$r^2 = \frac{r^2}{4} + h^2;$$

$$h^2 = r^2 - \frac{r^2}{4} = r^2\left(1 - \frac{1}{4}\right) = \frac{3r^2}{4}$$

$$\therefore h = \frac{\sqrt{3}}{2} r$$

$$A_T = \frac{r}{2} \bullet h = \frac{\sqrt{3}}{4} r^2$$

$$A_H = 6A_T = \frac{6\sqrt{3}}{4} r^2 = \frac{3\sqrt{3}}{2} r^2; \quad \text{but } r = \frac{d}{2}$$

$$\therefore A_H = \frac{3\sqrt{3}}{2}\left(\frac{d}{2}\right)^2 = \frac{3\sqrt{3}}{8} d^2$$

And $d = \sqrt{\frac{8A_H}{3\sqrt{3}}} = 2\sqrt{\frac{2A_H}{3\sqrt{3}}} \cong 1.2408\sqrt{A_H}$

This is the same result we obtained earlier when we set n=6 in the general formula.